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Abstract

We study the pure braid groups $P_n(\mathbb{R}P^2)$ of the real projective plane $\mathbb{R}P^2$, and in particular the possible splitting of the Fadell-Neuwirth short exact sequence $1 \longrightarrow P_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \hookrightarrow P_{n+m}(\mathbb{R}P^2) \xrightarrow{p_*} P_n(\mathbb{R}P^2) \longrightarrow 1$, where $n \geq 2$ and $m \geq 1$, and p_* is the homomorphism which corresponds geometrically to forgetting the last m strings. This problem is equivalent to that of the existence of a section for the associated fibration $p\colon F_{n+m}(\mathbb{R}P^2) \longrightarrow F_n(\mathbb{R}P^2)$ of configuration spaces. Van Buskirk proved in 1966 that p and p_* admit a section if $n = 2$ and $m = 1$. Our main result in this paper is to prove that there is no section if $n \geq 3$. As a corollary, it follows that $n = 2$ and $m = 1$ are the only values for which a section exists. As part of the proof, we derive a presentation of $P_n(\mathbb{R}P^2)$: this appears to be the first time that such a presentation has been given in the literature.

1 Introduction

Braid groups of the plane were defined by Artin in 1925 [A1], and further studied in [A2, A3]. They were later generalised using the following definition due to Fox [FoN]. Let M be a compact, connected surface, and let $n \in \mathbb{N}$. We denote the set of all ordered n-tuples of distinct points of M, known as the n^{th} *configuration space of M*, by:

$$F_n(M) = \left\{ (p_1, \ldots, p_n) \;\middle|\; p_i \in M \text{ and } p_i \neq p_j \text{ if } i \neq j \right\}.$$

Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [CG, FH] for example.

The symmetric group S_n on n letters acts freely on $F_n(M)$ by permuting coordinates. The corresponding quotient will be denoted by $D_n(M)$. Notice that $F_n(M)$ is a regular covering of $D_n(M)$. The n^{th} *pure braid group* $P_n(M)$ (respectively the n^{th} *braid group* $B_n(M)$) is defined to be the fundamental group of $F_n(M)$ (respectively of $D_n(M)$). If $m \in \mathbb{N}$, then we may define a homomorphism $p_*\colon P_{n+m}(M) \longrightarrow P_n(M)$ induced by the projection $p\colon F_{n+m}(M) \longrightarrow F_n(M)$ defined by $p((x_1,\ldots,x_n,\ldots,x_{n+m})) = (x_1,\ldots,x_n)$. Representing $P_{n+m}(M)$ geometrically as a collection of $n+m$ strings, p_* corresponds to forgetting the last m strings. **We adopt the convention, that unless explicitly stated, all homomorphisms $P_{n+m}(M) \longrightarrow P_n(M)$ in the text will be this one**.

If M is without boundary, Fadell and Neuwirth study the map p, and show ([FaN, Theorem 3]) that it is a locally-trivial fibration. The fibre over a point $(x_1,\ldots,x_n)$ of the base space is $F_m(M \setminus \{x_1,\ldots,x_n\})$ which we consider to be a subspace of the total space via the map $i\colon F_m(M \setminus \{x_1,\ldots,x_n\}) \longrightarrow F_n(M)$ defined by $i((y_1,\ldots,y_m)) = (x_1,\ldots,x_n,y_1,\ldots,y_m)$. Applying the associated long exact sequence in homotopy, we obtain the *pure braid group short exact sequence of Fadell and Neuwirth*:

$$1 \longrightarrow P_m(M \setminus \{x_1,\ldots,x_n\}) \xrightarrow{i_*} P_{n+m}(M) \xrightarrow{p_*} P_n(M) \longrightarrow 1, \qquad (\mathbf{PBS})$$

where $n \geq 3$ if M is the sphere $\mathbb{S}^2$ [Fa, FVB], $n \geq 2$ if M is the real projective plane $\mathbb{R}P^2$ [VB], and $n \geq 1$ otherwise [FaN], and where i_* and p_* are the homomorphisms induced by the maps i and p respectively. The sequence also exists for the classical pure braid group P_n, where M is the 2-disc $\mathbb{D}^2$ (or the plane). The short exact sequence (**PBS**) has been widely studied, and may be employed for example to determine presentations of $P_n(M)$ (see Section 2), its centre, and possible torsion. It was also used in recent work on the structure of the mapping class groups [PR] and on Vassiliev invariants for surface braids [GMP].

The decomposition of P_n as a repeated semi-direct product of free groups (known as the 'combing' operation) is the principal result of Artin's classical theory of braid groups [A2], and allows one to obtain normal forms and to solve the word problem. More recently, it was used by Falk and Randell to study the lower central series and the residual nilpotence of P_n [FR], and by Rolfsen and Zhu to prove that P_n is bi-orderable [RZ].

The problem of deciding whether such a decomposition exists for surface braid groups is thus fundamental. This was indeed a recurrent and central question during the foundation of the theory and its subsequent development during the 1960's [Fa, FaN, FVB, VB, Bi]. If the fibre of the fibration is an Eilenberg-MacLane space then the existence of a section for p_* is equivalent to that of a cross-section for p [Ba, Wh] (cf. [GG2]). But with the exception of the construction of sections in certain cases (for the sphere [Fa] and the torus [Bi]), no progress on the possible splitting of (**PBS**) was recorded for nearly forty years. In the case of orientable surfaces without boundary of genus at least two, the question of the splitting of (**PBS**) which was posed explicitly by Birman in 1969 [Bi], was finally resolved by the authors, the answer being positive if and only if $n = 1$ [GG1].

In this paper, we study the braid groups of $\mathbb{R}P^2$, in particular the splitting of the sequence (**PBS**), and the existence of a section for the fibration p. These groups were first studied by Van Buskirk [VB], and more recently by Wang [Wa]. Clearly $P_1(\mathbb{R}P^2) = B_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$. Van Buskirk showed that $P_2(\mathbb{R}P^2)$ is isomorphic to the quaternion group

$\mathcal{Q}_8$, $B_2(\mathbb{R}P^2)$ is a generalised quaternion group of order 16, and for $n > 2$, $P_n(\mathbb{R}P^2)$ and $B_n(\mathbb{R}P^2)$ are infinite. He also proved that these groups have elements of finite order (including one of order $2n$ in $B_n(\mathbb{R}P^2)$). The torsion elements (although not their orders) of $B_n(\mathbb{R}P^2)$ were characterised by Murasugi [M]. In [GG2], we showed that for $n \geq 2$, $B_n(\mathbb{R}P^2)$ has an element of order ℓ if and only if ℓ divides $4n$ or $4(n-1)$, and that $P_n(\mathbb{R}P^2)$ has torsion exactly 2 and 4. With respect to the splitting problem, Van Buskirk showed that for all $n \geq 2$, neither the fibration $p\colon F_n(\mathbb{R}P^2) \longrightarrow F_1(\mathbb{R}P^2)$ nor the homomorphism $p_*\colon P_n(\mathbb{R}P^2) \longrightarrow P_1(\mathbb{R}P^2)$ admit a cross-section (for p, this is a manifestation of the fixed point property of $\mathbb{R}P^2$), but that the fibration $p\colon F_3(\mathbb{R}P^2) \longrightarrow F_2(\mathbb{R}P^2)$ admits a cross-section, and hence so does the corresponding homomorphism. It follows from (**PBS**) that $P_3(\mathbb{R}P^2)$ is isomorphic to a semi-direct product of $\pi_1(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, which is a free group $\mathbb{F}_2$ of rank 2, by $P_2(\mathbb{R}P^2)$ which as we mentioned, is isomorphic to $\mathcal{Q}_8$ (see [GG2] for an explicit algebraic section). This fact will be used in the proof of Proposition 5 (see Section 3). Although there is no relation with the braid groups of the sphere, it is a curious fact that the commutator subgroup of $B_4(\mathbb{S}^2)$ is isomorphic to a semi-direct product of $\mathcal{Q}_8$ by $\mathbb{F}_2$ [GG4]. In fact $B_n(\mathbb{S}^2)$ possesses subgroups isomorphic to $\mathcal{Q}_8$ if and only if $n \geq 4$ is even [GG3].

In [GG2], we determined the homotopy type of the universal covering space of $F_n(\mathbb{R}P^2)$. From this, we were able to deduce the higher homotopy groups of $F_n(\mathbb{R}P^2)$. Using coincidence theory, we then showed that for $n = 2, 3$ and $m \geq 4 - n$, neither the fibration nor the short exact sequence (**PBS**) admit a section. More precisely:

Theorem 1 ([GG2]). *Let $r \geq 4$ and $n = 2, 3$. Then:*
(a) the fibration $p\colon F_r(\mathbb{R}P^2) \longrightarrow F_n(\mathbb{R}P^2)$ does not admit a cross-section.
(b) the Fadell-Neuwirth pure braid group short exact sequence :

$$1 \longrightarrow P_{r-n}(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \xrightarrow{i_*} P_r(\mathbb{R}P^2) \xrightarrow{p_*} P_n(\mathbb{R}P^2) \longrightarrow 1$$

does not split.

Apart from Van Buskirk's results for $F_n(\mathbb{R}P^2) \longrightarrow F_1(\mathbb{R}P^2)$ and $F_3(\mathbb{R}P^2) \longrightarrow F_2(\mathbb{R}P^2)$ (published in 1966), no other results are known concerning the splitting of (**PBS**) for the pure braid groups of $\mathbb{R}P^2$. The question is posed explicitly in the case $r = n + 1$ on page 97 of [VB]. In this paper, we give a complete answer. The main theorem is:

Theorem 2. *For all $n \geq 3$ and $m \geq 1$, the Fadell-Neuwirth pure braid group short exact sequence (***PBS***):*

$$1 \longrightarrow P_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \longrightarrow P_{n+m}(\mathbb{R}P^2) \xrightarrow{p_*} P_n(\mathbb{R}P^2) \longrightarrow 1$$

does not split, and the fibration $p\colon F_{n+m}(\mathbb{R}P^2) \longrightarrow F_n(\mathbb{R}P^2)$ does not admit a section.

Taking into account Van Buskirk's results and Theorem 1, we deduce immediately the following corollary:

Corollary 3. *If $m, n \in \mathbb{N}$, the homomorphism $p_*\colon P_{n+m}(\mathbb{R}P^2) \longrightarrow P_n(\mathbb{R}P^2)$ and the fibration $p\colon F_{n+m}(\mathbb{R}P^2) \longrightarrow F_n(\mathbb{R}P^2)$ admit a section if and only if $n = 2$ and $m = 1$.* □

In other words, Van Buskirk's values ($n = 2$ and $m = 1$) are the only ones for which a section exists (both on the geometric and the algebraic level). The splitting problem for non-orientable surfaces without boundary and of higher genus is the subject of work in progress [GG5]. In the case of the Klein bottle, the existence of a non-vanishing vector field implies that there always exists a section, both geometric and algebraic (cf. [FaN]).

This paper is organised as follows. In Section 2, we start by determining a presentation of $P_n(\mathbb{R}P^2)$ (Theorem 4). To the best of our knowledge, surprisingly this appears to be the first such presentation in the literature (although Van Buskirk gave a presentation of $B_n(\mathbb{R}P^2)$).

In order to prove Theorem 2, we argue by contradiction, and suppose that there exists some $n \geq 3$ for which a section occurs. As we indicate in Section 4, it then suffices to study the case $m = 1$. The general strategy of the proof of Theorem 2 is based on the following remark: if H is any normal subgroup of $P_{n+1}(\mathbb{R}P^2)$ contained in $\operatorname{Ker}(p_*)$, the quotiented short exact sequence $1 \longrightarrow \operatorname{Ker}(p_*)/H \hookrightarrow P_{n+1}(\mathbb{R}P^2)/H \longrightarrow P_n(\mathbb{R}P^2) \longrightarrow 1$ must also split. In order to reach a contradiction, we seek such a subgroup H for which this short exact sequence does *not* split. However the choice of H needed to achieve this is extremely delicate: if H is too 'small', the structure of the quotient $P_{n+1}(\mathbb{R}P^2)/H$ remains complicated; on the other hand, if H is too 'big', we lose too much information and cannot reach a conclusion. Taking a variety of possible candidates for H, we observed in preliminary calculations that the line between the two is somewhat fine. If n is odd, we were able to show that the problem may be solved by taking the quotient $\operatorname{Ker}(p_*)/H$ to be Abelianisation of $\operatorname{Ker}(p_*)$ (which is a free Abelian group of rank n) modulo 2, which is isomorphic to the direct sum of n copies of $\mathbb{Z}_2$. However, this insufficient for n even.

With this in mind, in Section 3, we study the quotient of $P_{n+1}(\mathbb{R}P^2)$ by a certain normal subgroup L which is contained in $\operatorname{Ker}(p_*)$ in the case $m = 1$. A key step in the proof of Theorem 2 is Proposition 5 where we show that $\operatorname{Ker}(p_*)/L$ is isomorphic to $\mathbb{Z}^{n-1} \rtimes \mathbb{Z}$, the action being given by multiplication by -1. This facilitates the calculations in $P_{n+1}(\mathbb{R}P^2)/L$, whilst leaving just enough room for a contradiction. This is accomplished in Section 4 where we show that the following quotiented short exact sequence:

$$1 \longrightarrow \operatorname{Ker}(p_*)/L \longrightarrow P_{n+1}(\mathbb{R}P^2)/L \longrightarrow P_n(\mathbb{R}P^2) \longrightarrow 1$$

does not split.

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2 A presentation of $P_n(\mathbb{R}P^2)$

If $n \in \mathbb{N}$ and $\mathbb{D}^2 \subseteq \mathbb{R}P^2$ is a topological disc, the inclusion induces a (non-injective) homomorphism $\iota\colon B_n(\mathbb{D}^2) \longrightarrow B_n(\mathbb{R}P^2)$. If $\beta \in B_n(\mathbb{D}^2)$ then we shall denote its image $\iota(\beta)$ simply by β. For $1 \leq i < j \leq n$, we consider the following elements of $P_n(\mathbb{R}P^2)$:

$$B_{i,j} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_i,$$

where $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of $B_n(\mathbb{D}^2)$. The geometric braid corresponding to $B_{i,j}$ takes the i^{th} string once around the j^{th} string in the positive sense, with all other strings remaining vertical. For each $1 \leq k \leq n$, we define a generator ρ_k which is represented geometrically by a loop based at the k^{th} point and which goes round the twisted handle. These elements are illustrated in Figure 1 ($\mathbb{R}P^2$ minus a disc may be thought of as the union of a disc and a twisted handle).

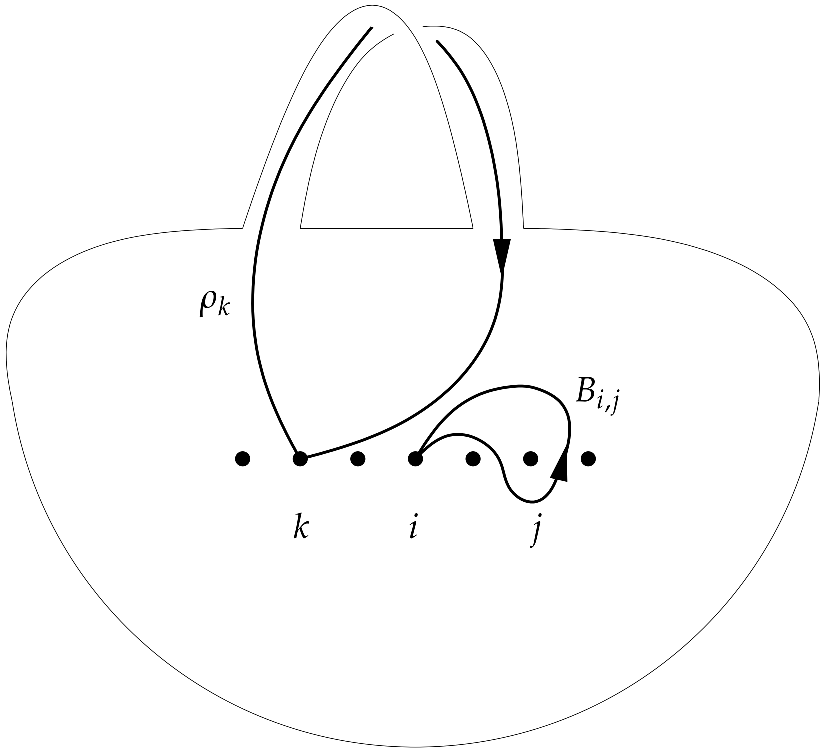

Figure 1: The generators $B_{i,j}$ and ρ_k of $P_n(\mathbb{R}P^2)$.

A presentation of $B_n(\mathbb{R}P^2)$ was first given by Van Buskirk in [VB]. Although presentations of braid groups of orientable and non-orientable surfaces have been the focus of several papers [Bi, S, GM, Be], we were not able to find an explicit presentation of $P_n(\mathbb{R}P^2)$ in the literature, so we derive one here.

Theorem 4. *Let $n \in \mathbb{N}$. The following constitutes a presentation of pure braid group $P_n(\mathbb{R}P^2)$:*

generators: *$B_{i,j}$, $1 \leq i < j \leq n$, and ρ_k, $1 \leq k \leq n$.*
relations:
(a) the Artin relations between the $B_{i,j}$ emanating from those of $P_n(\mathbb{D}^2)$:

$$B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j \\ B_{i,j}^{-1}B_{r,j}^{-1}B_{i,j}B_{r,j}B_{i,j} & \text{if } r<i=s<j \\ B_{s,j}^{-1}B_{i,j}B_{s,j} & \text{if } i=r<s<j \\ B_{s,j}^{-1}B_{r,j}^{-1}B_{s,j}B_{r,j}B_{i,j}B_{r,j}^{-1}B_{s,j}^{-1}B_{r,j}B_{s,j} & \text{if } r<i<s<j. \end{cases}$$

(b) for all $1 \leq i < j \leq n$, $\rho_i\rho_j\rho_i^{-1} = \rho_j^{-1}B_{i,j}^{-1}\rho_j^2$.
(c) for all $1 \leq i \leq n$, the 'surface relations' $\rho_i^2 = B_{1,i} \cdots B_{i-1,i}B_{i,i+1} \cdots B_{i,n}$.

(d) for all $1 \leq i < j \leq n$ and $1 \leq k \leq n$, $k \neq j$,

$$\rho_k B_{i,j} \rho_k^{-1} = \begin{cases} B_{i,j} & \text{if } j < k \text{ or } k < i \\ \rho_j^{-1} B_{i,j}^{-1} \rho_j & \text{if } k = i \\ \rho_j^{-1} B_{k,j}^{-1} \rho_j B_{k,j}^{-1} B_{i,j} B_{k,j} \rho_j^{-1} B_{k,j} \rho_j & \text{if } i < k < j. \end{cases}$$

Proof. We apply induction and standard results concerning the presentation of an extension (see Theorem 1, Chapter 13 of [J]).

First note that the given presentation is correct for $n = 1$ ($P_1(\mathbb{R}P^2) = \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$), and $n = 2$ ($P_2(\mathbb{R}P^2) \cong \mathcal{Q}_8$). So let $n \geq 2$, and suppose that $P_n(\mathbb{R}P^2)$ has the given presentation. Consider the corresponding Fadell-Neuwirth short exact sequence:

$$1 \longrightarrow \pi_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \longrightarrow P_{n+1}(\mathbb{R}P^2) \xrightarrow{p_*} P_n(\mathbb{R}P^2) \longrightarrow 1. \tag{1}$$

In order to retain the symmetry of the presentation, we take the free group $\operatorname{Ker}(p_*)$ to have the following one-relator presentation:

$$\left\langle \rho_{n+1}, B_{1,n+1}, \ldots, B_{n,n+1} \,\middle|\, \rho_{n+1}^2 = B_{1,n+1} \cdots B_{n,n+1} \right\rangle.$$

Together with these generators of $\operatorname{Ker}(p_*)$, the elements $B_{i,j}$, $1 \leq i < j \leq n$, and ρ_k, $1 \leq k \leq n$, of $P_{n+1}(\mathbb{R}P^2)$ (which are coset representatives of the generators of $P_n(\mathbb{R}P^2)$) form the required generating set of $P_{n+1}(\mathbb{R}P^2)$.

There are three classes of relations of $P_{n+1}(\mathbb{R}P^2)$ which are obtained as follows. The first consists of the single relation $\rho_{n+1}^2 = B_{1,n+1} \cdots B_{n,n+1}$ of $\operatorname{Ker}(p_*)$. The second class is obtained by rewriting the relators of the quotient in terms of the coset representatives, and expressing the corresponding element as a word in the generators of $\operatorname{Ker}(p_*)$. In this way, all of the relations of $P_n(\mathbb{R}P^2)$ lift directly to relations of $P_{n+1}(\mathbb{R}P^2)$, with the exception of the surface relations which become $\rho_i^2 = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n} B_{i,n+1}$ for all $1 \leq i \leq n$. Together with the relation of $\operatorname{Ker}(p_*)$, we obtain the complete set of surface relations (relations (c)) for $P_{n+1}(\mathbb{R}P^2)$.

The third class of relations is obtained by rewriting the conjugates of the generators of $\operatorname{Ker}(p_*)$ by the coset representatives in terms of the generators of $\operatorname{Ker}(p_*)$:

(i) For all $1 \leq i < j \leq n$ and $1 \leq l \leq n$,

$$B_{i,j} B_{l,n+1} B_{i,j}^{-1} = \begin{cases} B_{l,n+1} & \text{if } l < i \text{ or } j < l \\ B_{l,n+1}^{-1} B_{i,n+1}^{-1} B_{l,n+1} B_{i,n+1} B_{l,n+1} & \text{if } l = j \\ B_{j,n+1}^{-1} B_{l,n+1} B_{j,n+1} & \text{if } l = i \\ B_{j,n+1}^{-1} B_{i,n+1}^{-1} B_{j,n+1} B_{i,n+1} B_{l,n+1} B_{i,n+1}^{-1} B_{j,n+1}^{-1} B_{i,n+1} B_{j,n+1} & \text{if } i < l < j. \end{cases}$$

(ii) $B_{i,j} \rho_{n+1} B_{i,j}^{-1} = \rho_{n+1}$ for all $1 \leq i < j \leq n$.

(iii) $\rho_k \rho_{n+1} \rho_k^{-1} = \rho_{n+1}^{-1} B_{k,n+1}^{-1} \rho_{n+1}^2$ for all $1 \leq k \leq n$.

(iv) For all $1 \leq k, l \leq n$,

$$\rho_k B_{l,n+1} \rho_k^{-1} = \begin{cases} B_{l,n+1} & \text{if } k < l \\ \rho_{n+1}^{-1} B_{l,n+1}^{-1} \rho_{n+1} & \text{if } k = l \\ \rho_{n+1}^{-1} B_{k,n+1}^{-1} \rho_{n+1} B_{k,n+1}^{-1} B_{l,n+1} B_{k,n+1} \rho_{n+1}^{-1} B_{k,n+1} \rho_{n+1} & \text{if } l < k. \end{cases}$$

Then relations (a) for $P_{n+1}(\mathbb{R}P^2)$ are obtained from relations (a) for $P_n(\mathbb{R}P^2)$ and relations (i), relations (b) for $P_{n+1}(\mathbb{R}P^2)$ are obtained from relations (b) for $P_n(\mathbb{R}P^2)$ and relations (iii), and relations (d) for $P_{n+1}(\mathbb{R}P^2)$ are obtained from relations (d) for $P_n(\mathbb{R}P^2)$, relations (iv) and (ii). $\square$

For future use, it will be convenient at this point to record the following supplementary relations in $P_n(\mathbb{R}P^2)$ which are consequences of the presentation of Theorem 4. Let $1 \leq i < j \leq n$.

(I) The action of the ρ_i^{-1} on the ρ_j may be deduced from that of ρ_i: $\rho_i^{-1}\rho_j\rho_i = B_{i,j}^{-1}\rho_j$.

(II) By relations (b) and (d), we have:

$$\rho_i(B_{i,j}^{-1}\rho_j B_{i,j}\rho_j^{-1}B_{i,j})\rho_i^{-1} = \rho_j^{-1}B_{i,j}\rho_j \cdot \rho_j^{-1}B_{i,j}^{-1}\rho_j^2 \cdot \rho_j^{-1}B_{i,j}^{-1}\rho_j \cdot \rho_j^{-2}B_{i,j}\rho_j \cdot \rho_j^{-1}B_{i,j}^{-1}\rho_j = B_{i,j}^{-1}.$$

Hence $\rho_j B_{i,j}\rho_j^{-1} = B_{i,j}\rho_i^{-1}B_{i,j}^{-1}\rho_i B_{i,j}^{-1}$.

(III) From relations (b) and (I), we see that:

$$\rho_j\rho_i^{-1}\rho_j^{-1} = \rho_i^{-1}\rho_j^{-1}B_{i,j}^{-1}\rho_j\rho_i \cdot \rho_i^{-1} = \rho_j^{-1}B_{i,j} \cdot \rho_i^{-1}B_{i,j}^{-1}\rho_i \cdot B_{i,j}^{-1}\rho_j \cdot \rho_i^{-1} = B_{i,j},$$

so $\rho_j\rho_i\rho_j^{-1} = \rho_i B_{i,j}^{-1}$.

(IV) From relations (I) and (d), we obtain:

$$\rho_j^{-1}\rho_i\rho_j = \rho_i\rho_j^{-1}B_{i,j}\rho_j = \rho_i^2 B_{i,j}^{-1}\rho_i^{-1}.$$

3 A presentation of the quotient $P_{n+1}(\mathbb{R}P^2)/L$

For $n \geq 2$, we have the Fadell-Neuwirth short exact sequence (1) whose kernel $K = \operatorname{Ker}(p_*)$ is a free group of rank n with basis $\rho_{n+1}, B_{1,n+1}, B_{2,n+1}, \ldots, B_{n-1,n+1}$. We first introduce a subgroup L of K which is normal in $P_{n+1}(\mathbb{R}P^2)$, from which we shall be able to prove Theorem 2.

We define L to be the normal closure in $P_{n+1}(\mathbb{R}P^2)$ of the following elements:

(i) $[B_{i,n+1}, B_{j,n+1}]$, where $1 \leq i < j \leq n-1$, and

(ii) $[B_{i,n+1}, \rho_k]$, where $1 \leq i \leq n-1$ and $1 \leq k \leq n$.

The elements $[B_{i,n+1}, B_{j,n+1}]$ clearly belong to K. The presentation of $P_n(\mathbb{R}P^2)$ given by Theorem 4 implies that:

$$[B_{i,n+1}, \rho_k] = \begin{cases} 1 & \text{if } k < i \\ B_{i,n+1}\rho_{n+1}^{-1}B_{i,n+1}\rho_{n+1} & \text{if } k = i \\ B_{i,n+1}\rho_{n+1}^{-1}B_{k,n+1}^{-1}\rho_{n+1}B_{k,n+1}^{-1}B_{i,n+1}^{-1}B_{k,n+1}\rho_{n+1}^{-1}B_{k,n+1}\rho_{n+1} & \text{if } i < k \leq n. \end{cases}$$

Thus L is a (normal) subgroup of K.

Let $g\colon P_{n+1}(\mathbb{R}P^2) \longrightarrow P_{n+1}(\mathbb{R}P^2)/L$ denote the canonical projection. For $i = 1, \ldots, n-1$, let $A_i = g(B_{i,n+1})$. Apart from these elements, if x is a generator of $P_{n+1}(\mathbb{R}P^2)$, we shall not distinguish notationally between x and $g(x)$. The quotient $P_{n+1}(\mathbb{R}P^2)/L$ is generated by $\rho_1, \ldots, \rho_{n+1}$, $B_{i,j}$, $1 \leq i < j \leq n$, and $A_1, A_2, \ldots, A_{n-1}$ (we delete $B_{n,n+1}$ from the list using the surface relation $\rho_{n+1}^2 = A_1A_2\cdots A_{n-1}B_{n,n+1}$, so $B_{n,n+1} = A_{n-1}^{-1}\cdots A_2^{-1}A_1^{-1}\rho_{n+1}^2$).

A presentation of $P_{n+1}(\mathbb{R}P^2)/L$ may obtained from that of $P_{n+1}(\mathbb{R}P^2)$ by adding the relations arising from the elements of L. We list those relations which are relevant for our description of $P_{n+1}(\mathbb{R}P^2)/L$.

(a) The Artin relations between the $B_{i,j}$, $1 \leq i < j \leq n$.
(b) The relations of $P_n(\mathbb{R}P^2)$ between ρ_i, ρ_j, $1 \leq i < j \leq n$.
(c) The relations of $P_n(\mathbb{R}P^2)$ between $B_{i,j}$, $1 \leq i < j \leq n$ and ρ_k, $1 \leq k \leq n$.
The following two sets of relations arise from the definition of L:
(d) $A_i \rightleftharpoons A_j$, $1 \leq i < j \leq n-1$ (the symbol $\rightleftharpoons$ is used to mean that the given elements commute).
(e) $A_i \rightleftharpoons \rho_j$, $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$.
(f) The surface relations:

$$\rho_i^2 = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n} A_i \text{ for } i = 1, \ldots, n-1$$
$$\rho_n^2 = B_{1,n} B_{2,n} \cdots B_{n-1,n} \cdot A_{n-1}^{-1} \cdots A_2^{-1} A_1^{-1} \rho_{n+1}^2.$$

(g) For $i = 1, \ldots, n-1$, $\rho_{n+1} A_i \rho_{n+1}^{-1} = A_i^{-1}$ (since $\rho_{n+1}^{-1} A_i^{-1} \rho_{n+1} = \rho_i A_i \rho_i^{-1} = A_i$).
The following relations are implied by the above relations:
– for $1 \leq i < j \leq n-1$, $\rho_j A_i \rho_j^{-1} = \rho_{n+1}^{-1} A_j^{-1} \rho_{n+1} A_j^{-1} A_i A_j \rho_{n+1}^{-1} A_j \rho_{n+1}$ (both are equal to A_i).
– for $1 \leq i \leq n-1$, $\rho_{n+1} A_i \rho_{n+1}^{-1} = A_i \rho_i^{-1} A_i^{-1} \rho_i A_i^{-1}$ (both are equal to A_i^{-1})
(h) For $1 \leq j \leq n-1$,

$$\rho_i \rho_{n+1} \rho_i^{-1} = \rho_{n+1}^{-1} A_i^{-1} \rho_{n+1}^2 = A_i \rho_{n+1} = \rho_{n+1} A_i^{-1}.$$

From these relations, it follows that $\rho_i \rightleftharpoons \rho_{n+1}^2$ for $i = 1, \ldots, n-1$.
(i) $\rho_n \rho_{n+1} \rho_n^{-1} = \rho_{n+1}^{-1} B_{n,n+1}^{-1} \rho_{n+1}^2 = \rho_{n+1}^{-1} \rho_{n+1}^{-2} A_1 \cdots A_{n-1} \rho_{n+1}^2 = A_1^{-1} \cdots A_{n-1}^{-1} \rho_{n+1}^{-1}$. From this relation, it follows that $\rho_n \rho_{n+1}^2 \rho_n^{-1} = \rho_{n+1}^{-2}$.
For $i = 1 \ldots, n-1$, the following relations are implied by the above relations:

$$\rho_n A_i \rho_n^{-1} = \rho_{n+1}^{-1} B_{n,n+1}^{-1} \rho_{n+1} B_{n,n+1}^{-1} A_i B_{n,n+1} \rho_{n+1}^{-1} B_{n,n+1} \rho_{n+1} \text{ (both are equal to } A_i).$$

Proposition 5. *The quotient group K/L has a presentation of the form:*
generators: $A_1, \ldots, A_{n-1}, \rho_{n+1}$.
relations: $A_i \rightleftharpoons A_j$ *for* $1 \leq i < j \leq n-1$, *and* $\rho_{n+1} A_i \rho_{n+1}^{-1} = A_i^{-1}$ *for* $1 \leq i \leq n-1$.
In particular, K/L is isomorphic to $\mathbb{Z}^{n-1} \rtimes \mathbb{Z}$, the action being given by multiplication by -1.

Hence the other relations in $P_{n+1}(\mathbb{R}P^2)/L$ (which involve only elements from $P_n(\mathbb{R}P^2)$) do not add any further relations to the quotient K/L.

Proof of Proposition 5. Clearly $A_1, \ldots, A_{n-1}, \rho_{n+1}$ generate K/L, and from relations (d) and (g) of $P_{n+1}(\mathbb{R}P^2)/L$, they are subject to the given relations. Consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, hook] \arrow[d, "g|_K"] & P_{n+1}(\mathbb{R}P^2) \arrow[r, "p_*"] \arrow[d, "g"] & P_n(\mathbb{R}P^2) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & K/L \arrow[r, hook, "\iota"] & P_{n+1}(\mathbb{R}P^2)/L \arrow[r, "\overline{p}_*"] & P_n(\mathbb{R}P^2) \arrow[r] & 1,
\end{tikzcd} \tag{2}$$

where ι is the inclusion of K/L in $P_{n+1}(\mathbb{R}P^2)/L$, and $\overline{p}_*$ is the homomorphism induced by p_*. Let Γ be the group with presentation:

$$\Gamma = \left\langle \alpha_1, \ldots, \alpha_{n-1}, \rho \;\middle|\; \alpha_i \rightleftharpoons \alpha_j \text{ for } 1 \leq i < j \leq n-1, \text{ and } \rho \alpha_i \rho^{-1} = \alpha_i^{-1} \right\rangle.$$

So Γ is isomorphic to $\mathbb{Z}^{n-1} \rtimes \mathbb{Z}$, where the action is given by multiplication by -1. The map $f\colon \Gamma \longrightarrow K/L$ defined on the generators of Γ by $f(\alpha_i) = A_i$ for $i = 1, \ldots, n-1$, and $f(\rho) = \rho_{n+1}$, extends to a surjective homomorphism. We claim that f is an isomorphism, which will prove the proposition. To prove the claim, it suffices to show that $\iota \circ f$ is injective. Let $w \in \operatorname{Ker}(\iota \circ f)$. Then we may write w uniquely in the form $w = \rho^{m_0}\alpha_1^{m_1} \cdots \alpha_{n-1}^{m_{n-1}}$, where $m_0, m_1, \ldots, m_{n-1} \in \mathbb{Z}$, and so $\iota \circ f(w) = \rho_{n+1}^{m_0} A_1^{m_1} \cdots A_{n-1}^{m_{n-1}} = 1$ in $P_{n+1}(\mathbb{R}P^2)/L$.

Let $z = \rho_{n+1}^{m_0} B_{1,n+1}^{m_1} \cdots B_{n-1,n+1}^{m_{n-1}} \in P_{n+1}(\mathbb{R}P^2)$. Since $g(z) = \iota \circ f(w) = 1$, we must have $z \in L$. Now L is the normal closure in $P_{n+1}(\mathbb{R}P^2)$ of the following elements:

– $c_{j,k} = [B_{j,n+1}, B_{k,n+1}]$, where $1 \le j < k \le n-1$,

– $d_j = [B_{j,n+1}, \rho_j] = B_{j,n+1}\rho_{n+1}^{-1}B_{j,n+1}\rho_{n+1}$, where $1 \le j \le n-1$,

– $e_{j,k} = [B_{j,n+1}, \rho_k] = B_{j,n+1}\rho_{n+1}^{-1}B_{k,n+1}^{-1}\rho_{n+1}B_{k,n+1}^{-1}B_{j,n+1}^{-1}B_{k,n+1}\rho_{n+1}^{-1}B_{k,n+1}\rho_{n+1}$, where $1 \le j < k \le n$.

Hence z may be written as a product of conjugates of the elements $c_{j,k}$, d_j, $e_{j,k}$, and their inverses.

For $i = 1, \ldots, n-1$, let $\pi_i\colon P_{n+1}(\mathbb{R}P^2) \longrightarrow P_3(\mathbb{R}P^2, (p_i, p_n, p_{n+1}))$ be the projection obtained geometrically by forgetting all of the strings, with the exception of the i^{th}, n^{th} and $(n+1)^{\text{st}}$ strings (here $P_3(\mathbb{R}P^2, (p_i, p_n, p_{n+1}))$ denotes the fundamental group of $F_3(\mathbb{R}P^2)$ taking the basepoint to be (p_i, p_n, p_{n+1})). We interpret $P_3(\mathbb{R}P^2, (p_i, p_n, p_{n+1}))$ as the semi-direct product $\mathbb{F}_2(B_{i,n+1}, \rho_{n+1}) \rtimes P_2(\mathbb{R}P^2, (p_i, p_n))$ [VB]. Under π_i, the elements $c_{j,k}, d_j, e_{j,k}$ (for the allowed values of j and k) are all sent to the trivial element, with the exception of the two elements d_i and $e_{i,n}$. Set $h_i = \pi_i(d_i) = B_{i,n+1}\rho_{n+1}^{-1}B_{i,n+1}\rho_{n+1} \in \mathbb{F}_2(B_{i,n+1}, \rho_{n+1})$. Since $B_{i,n+1}B_{n,n+1} = \rho_{n+1}^2$ in $P_3(\mathbb{R}P^2, (p_i, p_n, p_{n+1}))$, we have $B_{n,n+1} = B_{i,n+1}^{-1}\rho_{n+1}^2$. Hence:

$$\begin{aligned}
\pi_i(e_{i,n}) =& B_{i,n+1}\rho_{n+1}^{-1}B_{n,n+1}^{-1}\rho_{n+1}B_{n,n+1}^{-1}B_{i,n+1}^{-1}B_{n,n+1}\rho_{n+1}^{-1}B_{n,n+1}\rho_{n+1}\\
=& B_{i,n+1}\rho_{n+1}^{-1}\rho_{n+1}^{-2}B_{i,n+1}\rho_{n+1}\rho_{n+1}^{-2}B_{i,n+1}B_{i,n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}^{2}\rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}^{2}\rho_{n+1}\\
=& B_{i,n+1}\rho_{n+1}^{-1}B_{i,n+1}\rho_{n+1} \cdot \rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}B_{i,n+1}^{-1}B_{i,n+1}\rho_{n+1} \cdot\\
& \rho_{n+1}^{-2}B_{i,n+1}\rho_{n+1}^{-1}B_{i,n+1}\rho_{n+1}\rho_{n+1}^{2} \cdot \rho_{n+1}^{-3}\rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}B_{i,n+1}^{-1}\rho_{n+1}^{3}\\
=& h_i \cdot \rho_{n+1}^{-1}B_{i,n+1}^{-1}h_i^{-1}B_{i,n+1}\rho_{n+1} \cdot \rho_{n+1}^{-2}h_i\rho_{n+1}^{2} \cdot \rho_{n+1}^{-3}h_i^{-1}\rho_{n+1}^{3}.
\end{aligned}$$

Thus $\pi_i(z)$ may be written as a product of conjugates in $P_3(\mathbb{R}P^2, (p_i, p_n, p_{n+1}))$ of $h_i^{\pm 1}$:

$$\pi_i(z) = \rho_{n+1}^{m_0}B_{i,n+1}^{m_i} = \prod_{j=1}^{l} w_j h_i^{\mu(j)} w_j^{-1}, \tag{3}$$

where $l \in \mathbb{N}$, $w_j \in P_3(\mathbb{R}P^2, (p_i, p_n, p_{n+1}))$, and $\mu(j) \in \{1, -1\}$. We claim that each $w_j h_i^{\mu(j)} w_j^{-1}$ is in fact a conjugate in $\mathbb{F}_2(B_{i,n+1}, \rho_{n+1})$ of $h_i^{\pm 1}$. This follows by studying the action of the generators ρ_i and ρ_n of $P_2(\mathbb{R}P^2, (p_i, p_n))$ on the basis of $\mathbb{F}_2(B_{i,n+1}, \rho_{n+1})$:

$$\begin{aligned}
\rho_i h_i \rho_i^{-1} =& \rho_i B_{i,n+1}\rho_{n+1}^{-1}B_{i,n+1}\rho_{n+1}\rho_i^{-1}\\
=& \rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1} \cdot \rho_{n+1}^{-2}B_{i,n+1}\rho_{n+1} \cdot \rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1} \cdot \rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}^{2}\\
=& \rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}^{2}\\
=& \rho_{n+1}^{-1}B_{i,n+1}^{-1} \cdot \rho_{n+1}^{-1}B_{i,n+1}^{-1}\rho_{n+1}B_{i,n+1}^{-1} \cdot B_{i,n+1}\rho_{n+1} = \rho_{n+1}^{-1}B_{i,n+1}^{-1}h_i^{-1}B_{i,n+1}\rho_{n+1},
\end{aligned}$$

and

$$\begin{aligned}
\rho_n h_i \rho_n^{-1} =& \rho_n B_{i,n+1} \rho_{n+1}^{-1} B_{i,n+1} \rho_{n+1} \rho_n^{-1}\\
=& \rho_{n+1}^{-1} B_{n,n+1}^{-1} \rho_{n+1} B_{n,n+1}^{-1} B_{i,n+1} B_{n,n+1} \rho_{n+1}^{-1} B_{n,n+1} \rho_{n+1} \cdot \rho_{n+1}^{-2} B_{n,n+1} \rho_{n+1} \cdot\\
& \rho_{n+1}^{-1} B_{n,n+1}^{-1} \rho_{n+1} B_{n,n+1}^{-1} B_{i,n+1} B_{n,n+1} \rho_{n+1}^{-1} B_{n,n+1} \rho_{n+1} \cdot \rho_{n+1}^{-1} B_{n,n+1}^{-1} \rho_{n+1}^{2}\\
=& \rho_{n+1}^{-1} B_{n,n+1}^{-1} \rho_{n+1} B_{n,n+1}^{-1} B_{i,n+1} B_{n,n+1} \rho_{n+1}^{-1} B_{i,n+1} B_{n,n+1} \rho_{n+1}\\
=& \rho_{n+1}^{-1} \rho_{n+1}^{-2} B_{i,n+1} \rho_{n+1} \rho_{n+1}^{-2} B_{i,n+1} B_{i,n+1} B_{i,n+1}^{-1} \rho_{n+1}^{2} \rho_{n+1}^{-1} B_{i,n+1} B_{i,n+1}^{-1} \rho_{n+1}^{2} \rho_{n+1}\\
=& \rho_{n+1}^{-3} B_{i,n+1} \rho_{n+1}^{-1} B_{i,n+1} \rho_{n+1} \rho_{n+1}^{3} = \rho_{n+1}^{-3} h_i \rho_{n+1}^{3},
\end{aligned}$$

again using the fact that $B_{i,n+1} B_{n,n+1} = \rho_{n+1}^2$ in $P_3(\mathbb{R}P^2, (p_i, p_n, p_{n+1}))$. Thus the w_j of equation (3) may be taken as belonging to $\mathbb{F}_2(B_{i,n+1}, \rho_{n+1})$. We now project $\mathbb{F}_2(B_{i,n+1}, \rho_{n+1})$ onto the Klein bottle group $\left\langle B_{i,n+1}, \rho_{n+1} \;\middle|\; \rho_{n+1}^{-1} B_{i,n+1} \rho_{n+1} = B_{i,n+1}^{-1} \right\rangle$ in the obvious manner. Since h_i belongs to the kernel of this projection, the right hand-side of equation (3) is sent to the trivial element, while the left hand-side is sent to $\rho_{n+1}^{m_0} B_{i,n+1}^{m_i}$. It follows that $m_0 = m_i = 0$ for all $i = 1, \ldots, n-1$. This proves the injectivity of $\iota \circ f$, and so completes the proof of the proposition. $\square$

4 Proof of Theorem 2

We are now ready to give the proof of the main theorem of the paper.

Proof of Theorem 2. Let $n \geq 3$. For $m \geq 1$, let $p_*^{(m)} \colon P_{n+m}(\mathbb{R}P^2) \longrightarrow P_n(\mathbb{R}P^2)$ denote the usual projection. Suppose first that $m \geq 2$, and consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & P_m(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \arrow[r, hook] \arrow[d, "\psi\left|_{P_m\left(\mathbb{R}P^2\setminus\{x_1,\ldots,x_n\}\right)}\right."] & P_{n+m}(\mathbb{R}P^2) \arrow[r, "p_*^{(m)}"] \arrow[d, "\psi"] & P_n(\mathbb{R}P^2) \arrow[r] \arrow[d, equal] & 1\\
1 \arrow[r] & P_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}) \arrow[r, hook] & P_{n+1}(\mathbb{R}P^2) \arrow[r, "p_*^{(1)}"] & P_n(\mathbb{R}P^2) \arrow[r] & 1,
\end{tikzcd}$$

where ψ is the homomorphism which forgets the last $m-1$ strings. If $p_*^{(m)}$ admits a section $s_*^{(m)}$ then $\psi \circ s_*^{(m)}$ is a section for $p_*^{(1)}$. In other words, if the upper short exact sequence splits then so does the lower one.

Since we shall be arguing for a contradiction, we are reduced to considering the case $m = 1$. Set $p_* = p_*^{(1)}$, and suppose that p_* admits a section which we shall denote by s_*. Consider the short exact sequence (2). Since p_* admits a section then so does $\overline{p}_*$; we denote its section by $\overline{s}_*$. So $\overline{p}_*(\rho_i) = \rho_i$ for $i = 1, \ldots, n$, and $\overline{p}_*(B_{i,j}) = B_{i,j}$ for $1 \leq i < j \leq n$ (recall that we do not distinguish notationally between the generators of $P_{n+1}(\mathbb{R}P^2)/L$ and the corresponding generators of $P_n(\mathbb{R}P^2)$). Thus we obtain:

$$\left.\begin{aligned}
\overline{s}_*(\rho_i) &= \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \cdot \rho_i \text{ for } i = 1, \ldots, n\\
\overline{s}_*(B_{i,j}) &= \rho_{n+1}^{\beta_{i,j,0}} A_1^{\beta_{i,j,1}} \cdots A_{n-1}^{\beta_{i,j,n-1}} \cdot B_{i,j} \text{ for } 1 \leq i < j \leq n,
\end{aligned}\right\} \tag{4}$$

where $\alpha_{i,k}, \beta_{i,j,k} \in \mathbb{Z}$. For $x \in \mathbb{Z}$, set

$$\varepsilon(x) = \begin{cases} 1 & \text{if } x \text{ is even}\\ -1 & \text{if } x \text{ is odd,} \end{cases} \qquad \text{and} \qquad \delta(x) = \begin{cases} 0 & \text{if } x \text{ is even}\\ -1 & \text{if } x \text{ is odd.} \end{cases}$$

Then $\varepsilon(x) = 2\delta(x)+1$, $\varepsilon(x)\delta(x) = -\delta(x)$, $\delta(x) = \delta(-x)$, $\varepsilon(x) = \varepsilon(-x)$ and for $i = 1,\ldots,n-1$ and $k \in \mathbb{Z}$, we have:

$$\begin{aligned}
\rho_{n+1}^{k} A_i \rho_{n+1}^{-k} &= A_i^{\varepsilon(k)}\\
\rho_i \rho_{n+1}^{k} \rho_i^{-1} &= \rho_{n+1}^{k} A_i^{\delta(k)}\\
\rho_i^{-1} \rho_{n+1}^{k} \rho_i &= \rho_{n+1}^{k} A_i^{-\delta(k)}\\
\rho_n \rho_{n+1}^{k} \rho_n^{-1} &= \rho_{n+1}^{-k} A_1^{-\delta(k)} \cdots A_{n-1}^{-\delta(k)}\\
\rho_n^{-1} \rho_{n+1}^{k} \rho_n &= \rho_{n+1}^{-k} A_1^{\delta(k)} \cdots A_{n-1}^{\delta(k)},
\end{aligned}$$

using the relations of $P_{n+1}(\mathbb{R}P^2)/L$ given in Section 3.

We now calculate the images in $P_{n+1}(\mathbb{R}P^2)/L$ by $\overline{s}_*$ of the following relations of $P_n(\mathbb{R}P^2)$. This will allow us to obtain information about the coefficients defined in equation (4).

(a) We start with the relation $\underline{\rho_j\rho_i\rho_j^{-1} = \rho_i B_{i,j}^{-1}}$ in $P_n(\mathbb{R}P^2)$, where $1 \le i < j \le n-1$.

$$\begin{aligned}
\overline{s}_*(\rho_i B_{i,j}^{-1}) =& \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \rho_i \cdot B_{i,j}^{-1} A_{n-1}^{-\beta_{i,j,n-1}} \cdots A_1^{-\beta_{i,j,1}} \rho_{n+1}^{-\beta_{i,j,0}}\\
=& \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \rho_i \rho_{n+1}^{-\beta_{i,j,0}} B_{i,j}^{-1} A_{n-1}^{-\varepsilon(\beta_{i,j,0})\beta_{i,j,n-1}} \cdots A_1^{-\varepsilon(\beta_{i,j,0})\beta_{i,j,1}}\\
=& \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \rho_{n+1}^{-\beta_{i,j,0}} A_i^{\delta(\beta_{i,j,0})} \rho_i A_{n-1}^{-\varepsilon(\beta_{i,j,0})\beta_{i,j,n-1}} \cdots A_1^{-\varepsilon(\beta_{i,j,0})\beta_{i,j,1}} B_{i,j}^{-1}\\
=& \rho_{n+1}^{\alpha_{i,0}-\beta_{i,j,0}} A_1^{\varepsilon(\beta_{i,j,0})\alpha_{i,1}} \cdots A_{n-1}^{\varepsilon(\beta_{i,j,0})\alpha_{i,n-1}} A_i^{\delta(\beta_{i,j,0})}\cdot\\
& A_{n-1}^{-\varepsilon(\beta_{i,j,0})\beta_{i,j,n-1}} \cdots A_1^{-\varepsilon(\beta_{i,j,0})\beta_{i,j,1}} \rho_i B_{i,j}^{-1}\\
=& \rho_{n+1}^{\alpha_{i,0}-\beta_{i,j,0}} A_1^{\varepsilon(\beta_{i,j,0})(\alpha_{i,1}-\beta_{i,j,1})} \cdots A_i^{\varepsilon(\beta_{i,j,0})(\alpha_{i,i}-\beta_{i,j,i})+\delta(\beta_{i,j,0})}\cdot\\
& \cdots A_{n-1}^{\varepsilon(\beta_{i,j,0})(\alpha_{i,n-1}-\beta_{i,j,n-1})} \rho_i B_{i,j}^{-1}.
\end{aligned}$$

$$\begin{aligned}
\overline{s}_*(\rho_j\rho_i\rho_j^{-1}) =& \rho_{n+1}^{\alpha_{j,0}} A_1^{\alpha_{j,1}} \cdots A_{n-1}^{\alpha_{j,n-1}} \rho_j \cdot \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \cdot \rho_i \cdot \rho_j^{-1} A_{n-1}^{-\alpha_{j,n-1}} \cdots A_1^{-\alpha_{j,1}} \rho_{n+1}^{-\alpha_{j,0}}\\
=& \rho_{n+1}^{\alpha_{j,0}+\alpha_{i,0}} A_1^{\varepsilon(\alpha_{i,0})\alpha_{j,1}} \cdots A_{n-1}^{\varepsilon(\alpha_{i,0})\alpha_{j,n-1}} A_j^{\delta(\alpha_{i,0})} \rho_j A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \rho_{n+1}^{-\alpha_{j,0}} A_i^{\delta(\alpha_{j,0})} \rho_i\cdot\\
& A_j^{-\delta(\alpha_{j,0})} \rho_j^{-1} A_{n-1}^{-\varepsilon(\alpha_{j,0})\alpha_{j,n-1}} \cdots A_1^{-\varepsilon(\alpha_{j,0})\alpha_{j,1}}\\
=& \rho_{n+1}^{\alpha_{i,0}} A_1^{\varepsilon(\alpha_{j,0})\varepsilon(\alpha_{i,0})\alpha_{j,1}} \cdots A_{n-1}^{\varepsilon(\alpha_{j,0})\varepsilon(\alpha_{i,0})\alpha_{j,n-1}} A_j^{\varepsilon(\alpha_{j,0})\delta(\alpha_{i,0})} A_j^{\delta(\alpha_{j,0})}\cdot\\
& A_1^{\varepsilon(\alpha_{j,0})\alpha_{i,1}} \cdots A_{n-1}^{\varepsilon(\alpha_{j,0})\alpha_{i,n-1}} A_i^{\delta(\alpha_{j,0})} A_j^{-\delta(\alpha_{j,0})} A_{n-1}^{-\varepsilon(\alpha_{j,0})\alpha_{j,n-1}} \cdots A_1^{-\varepsilon(\alpha_{j,0})\alpha_{j,1}} \rho_j\rho_i\rho_j^{-1}\\
=& \rho_{n+1}^{\alpha_{i,0}} A_1^{\varepsilon(\alpha_{j,0})\left(\alpha_{j,1}(\varepsilon(\alpha_{i,0})-1)+\alpha_{i,1}\right)} \cdots A_i^{\varepsilon(\alpha_{j,0})\left(\alpha_{j,i}(\varepsilon(\alpha_{i,0})-1)+\alpha_{i,i}\right)+\delta(\alpha_{j,0})}\cdot\\
& \cdots A_j^{\varepsilon(\alpha_{j,0})\left(\alpha_{j,j}(\varepsilon(\alpha_{i,0})-1)+\alpha_{i,j}+\delta(\alpha_{i,0})\right)} \cdots A_{n-1}^{\varepsilon(\alpha_{j,0})\left(\alpha_{j,n-1}(\varepsilon(\alpha_{i,0})-1)+\alpha_{i,n-1}\right)} \rho_j\rho_i\rho_j^{-1}.
\end{aligned}$$

Comparing coefficients in K/L, we obtain:

$$\begin{aligned}
&\beta_{i,j,0} = 0, \text{ so } \varepsilon(\beta_{i,j,0}) = 1 \text{ and } \delta(\beta_{i,j,0}) = 0 \text{ for all } 1 \le i < j \le n-1 && (5)\\
&\varepsilon(\alpha_{j,0})\alpha_{j,k}(\varepsilon(\alpha_{i,0})-1) + \alpha_{i,k}(\varepsilon(\alpha_{j,0})-1) = -\beta_{i,j,k} \text{ for all } k = 1,\ldots,n-1,\ k \neq i,j\\
&\varepsilon(\alpha_{j,0})\alpha_{j,i}(\varepsilon(\alpha_{i,0})-1) + \alpha_{i,i}(\varepsilon(\alpha_{j,0})-1) + \delta(\alpha_{j,0}) = -\beta_{i,j,i}\\
&\varepsilon(\alpha_{j,0})\alpha_{j,j}(\varepsilon(\alpha_{i,0})-1) + \alpha_{i,j}(\varepsilon(\alpha_{j,0})-1) + \varepsilon(\alpha_{j,0})\delta(\alpha_{i,0}) = -\beta_{i,j,j}.
\end{aligned}$$

In particular, the coefficient $\beta_{i,j,0}$ of ρ_{n+1} in $\overline{s}_*(B_{i,j})$ is zero. Also, since $\varepsilon(x)-1$ is even for all $x \in \mathbb{Z}$, $\beta_{i,j,k} \equiv 0 \pmod 2$ for all $k \neq i, j$, and for all $1 \leq i < j \leq n-1$,

$$\beta_{i,j,i} \equiv \delta(\alpha_{j,0}) \pmod 2 \tag{6}$$
$$\beta_{i,j,j} \equiv \delta(\alpha_{i,0}) \pmod 2. \tag{7}$$

(b) Now consider the relation $\underline{\rho_n\rho_i\rho_n^{-1} = \rho_i B_{i,n}^{-1}}$ in $P_n(\mathbb{R}P^2)$, where $1 \leq i \leq n-1$.

$$\begin{aligned}
\overline{s}_*(\rho_i B_{i,n}^{-1}) =& \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \rho_i \cdot B_{i,n}^{-1} A_{n-1}^{-\beta_{i,n,n-1}} \cdots A_1^{-\beta_{i,n,1}} \rho_{n+1}^{-\beta_{i,n,0}} \\
=& \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \rho_i \rho_{n+1}^{-\beta_{i,n,0}} B_{i,n}^{-1} A_{n-1}^{-\varepsilon(\beta_{i,n,0})\beta_{i,n,n-1}} \cdots A_1^{-\varepsilon(\beta_{i,n,0})\beta_{i,n,1}} \\
=& \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \rho_{n+1}^{-\beta_{i,n,0}} A_i^{\delta(\beta_{i,n,0})} \rho_i A_{n-1}^{-\varepsilon(\beta_{i,n,0})\beta_{i,n,n-1}} \cdots A_1^{-\varepsilon(\beta_{i,n,0})\beta_{i,n,1}} B_{i,n}^{-1} \\
=& \rho_{n+1}^{\alpha_{i,0}-\beta_{i,n,0}} A_1^{\varepsilon(\beta_{i,n,0})\alpha_{i,1}} \cdots A_{n-1}^{\varepsilon(\beta_{i,n,0})\alpha_{i,n-1}} A_i^{\delta(\beta_{i,n,0})} \cdot \\
& A_{n-1}^{-\varepsilon(\beta_{i,n,0})\beta_{i,n,n-1}} \cdots A_1^{-\varepsilon(\beta_{i,n,0})\beta_{i,n,1}} \rho_i B_{i,n}^{-1} \\
=& \rho_{n+1}^{\alpha_{i,0}-\beta_{i,n,0}} A_1^{\varepsilon(\beta_{i,n,0})(\alpha_{i,1}-\beta_{i,n,1})} \cdots A_i^{\varepsilon(\beta_{i,n,0})(\alpha_{i,i}-\beta_{i,n,i})+\delta(\beta_{i,n,0})} \cdot \\
& \cdots A_{n-1}^{\varepsilon(\beta_{i,n,0})(\alpha_{i,n-1}-\beta_{i,n,n-1})} \rho_i B_{i,n}^{-1}.
\end{aligned}$$

$$\begin{aligned}
\overline{s}_*(\rho_n\rho_i\rho_n^{-1}) =& \rho_{n+1}^{\alpha_{n,0}} A_1^{\alpha_{n,1}} \cdots A_{n-1}^{\alpha_{n,n-1}} \rho_n \cdot \rho_{n+1}^{\alpha_{i,0}} A_1^{\alpha_{i,1}} \cdots A_{n-1}^{\alpha_{i,n-1}} \rho_i \cdot \rho_n^{-1} A_{n-1}^{-\alpha_{n,n-1}} \cdots A_1^{-\alpha_{n,1}} \rho_{n+1}^{-\alpha_{n,0}} \\
=& \rho_{n+1}^{\alpha_{n,0}-\alpha_{i,0}} A_1^{\varepsilon(\alpha_{i,0})\alpha_{n,1}} \cdots A_{n-1}^{\varepsilon(\alpha_{i,0})\alpha_{n,n-1}} A_1^{-\delta(\alpha_{i,0})} \cdots A_{n-1}^{-\delta(\alpha_{i,0})} \rho_n \rho_{n+1}^{\alpha_{n,0}} \cdot \\
& A_1^{\varepsilon(\alpha_{n,0})\alpha_{i,1}} \cdots A_{n-1}^{\varepsilon(\alpha_{n,0})\alpha_{i,n-1}} A_i^{\delta(\alpha_{n,0})} \rho_i A_1^{\delta(\alpha_{n,0})} \cdots A_{n-1}^{\delta(\alpha_{n,0})} \rho_n^{-1} \cdot \\
& A_{n-1}^{-\varepsilon(\alpha_{n,0})\alpha_{n,n-1}} \cdots A_1^{-\varepsilon(\alpha_{n,0})\alpha_{n,1}} \\
=& \rho_{n+1}^{-\alpha_{i,0}} A_1^{\varepsilon(\alpha_{n,0})\varepsilon(\alpha_{i,0})\alpha_{n,1}} \cdots A_{n-1}^{\varepsilon(\alpha_{n,0})\varepsilon(\alpha_{i,0})\alpha_{n,n-1}} A_1^{-\varepsilon(\alpha_{n,0})\delta(\alpha_{i,0})} \cdots A_{n-1}^{-\varepsilon(\alpha_{n,0})\delta(\alpha_{i,0})} \cdot \\
& A_1^{-\delta(\alpha_{n,0})} \cdots A_{n-1}^{-\delta(\alpha_{n,0})} A_1^{\varepsilon(\alpha_{n,0})\alpha_{i,1}} \cdots A_{n-1}^{\varepsilon(\alpha_{n,0})\alpha_{i,n-1}} A_i^{\delta(\alpha_{n,0})} A_1^{\delta(\alpha_{n,0})} \cdots A_{n-1}^{\delta(\alpha_{n,0})} \cdot \\
& A_{n-1}^{-\varepsilon(\alpha_{n,0})\alpha_{n,n-1}} \cdots A_1^{-\varepsilon(\alpha_{n,0})\alpha_{n,1}} \rho_n\rho_i\rho_n^{-1}.
\end{aligned}$$

Comparing coefficients in K/L, we obtain:

$$\beta_{i,n,0} = 2\alpha_{i,0}, \text{ so } \beta_{i,n,0} \text{ is even, } \varepsilon(\beta_{i,n,0}) = 1 \text{ and } \delta(\beta_{i,n,0}) = 0 \tag{8}$$
$$\varepsilon(\alpha_{n,0})\alpha_{n,k}(\varepsilon(\alpha_{i,0})-1) + \alpha_{i,k}(\varepsilon(\alpha_{n,0})-1) - \varepsilon(\alpha_{n,0})\delta(\alpha_{i,0}) = -\beta_{i,n,k} \text{ for } k = 1, \ldots, n-1, k \neq i$$
$$\varepsilon(\alpha_{n,0})\alpha_{n,i}(\varepsilon(\alpha_{i,0})-1) + \alpha_{i,i}(\varepsilon(\alpha_{n,0})-1) - \varepsilon(\alpha_{n,0})\delta(\alpha_{i,0}) + \delta(\alpha_{n,0}) = -\beta_{i,n,i}.$$

In particular, the coefficient $\beta_{i,n,0}$ of ρ_{n+1} in $\overline{s}_*(B_{i,n})$ is even. Further:

$$\begin{aligned}
&\beta_{i,n,k} \equiv \delta(\alpha_{i,0}) \pmod 2 \text{ for all } k \neq i \\
&\beta_{i,n,i} \equiv \delta(\alpha_{i,0}) + \delta(\alpha_{n,0}) \pmod 2 \text{ for all } 1 \leq i \leq n-1.
\end{aligned} \tag{9}$$

(c) Consider the relation $\underline{\rho_i^2 = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n}}$ in $P_n(\mathbb{R}P^2)$, where $1 \leq i \leq n -$

1. Using equations (5) and (8), we see that:

$$\begin{aligned}\overline{s}_*(B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n-1}B_{i,n}) =& A_1^{\beta_{1,i,1}}\cdots A_{n-1}^{\beta_{1,i,n-1}}B_{1,i}\cdots A_1^{\beta_{i-1,i,1}}\cdots A_{n-1}^{\beta_{i-1,i,n-1}}B_{i-1,i}\cdot\\ & A_1^{\beta_{i,i+1,1}}\cdots A_{n-1}^{\beta_{i,i+1,n-1}}B_{i,i+1}\cdots A_1^{\beta_{i,n-1,1}}\cdots A_{n-1}^{\beta_{i,n-1,n-1}}\cdot\\ & B_{i,n-1}\rho_{n+1}^{2\alpha_{i,0}}A_1^{\beta_{i,n,1}}\cdots A_{n-1}^{\beta_{i,n,n-1}}B_{i,n}\\ =&\rho_{n+1}^{2\alpha_{i,0}}A_1^{\beta_{1,i,1}+\cdots+\beta_{i-1,i,1}+\beta_{i,i+1,1}+\cdots+\beta_{i,n-1,1}+\beta_{i,n,1}}\cdot\\ & \cdots A_{n-1}^{\beta_{1,i,n-1}+\cdots+\beta_{i-1,i,n-1}+\beta_{i,i+1,n-1}+\cdots+\beta_{i,n-1,n-1}+\beta_{i,n,n-1}}\cdot\\ & B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n-1}B_{i,n}\\ =&\rho_{n+1}^{2\alpha_{i,0}}A_1^{\beta_{1,i,1}+\cdots+\beta_{i-1,i,1}+\beta_{i,i+1,1}+\cdots+\beta_{i,n-1,1}+\beta_{i,n,1}}\cdot\\ & \cdots A_i^{\beta_{1,i,i}+\cdots+\beta_{i-1,i,i}+\beta_{i,i+1,i}+\cdots+\beta_{i,n-1,i}+\beta_{i,n,i}-1}\cdot\\ & \cdots A_{n-1}^{\beta_{1,i,n-1}+\cdots+\beta_{i-1,i,n-1}+\beta_{i,i+1,n-1}+\cdots+\beta_{i,n-1,n-1}+\beta_{i,n,n-1}}\rho_i^2,\end{aligned}$$

using the relation $B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n-1}B_{i,n}A_i=\rho_i^2$ in $P_{n+1}(\mathbb{R}P^2)/L$.

$$\begin{aligned}\overline{s}_*(\rho_i^2) =&\rho_{n+1}^{\alpha_{i,0}}A_1^{\alpha_{i,1}}\cdots A_{n-1}^{\alpha_{i,n-1}}\rho_i\cdot\rho_{n+1}^{\alpha_{i,0}}A_1^{\alpha_{i,1}}\cdots A_{n-1}^{\alpha_{i,n-1}}\rho_i\\ =&\rho_{n+1}^{2\alpha_{i,0}}A_1^{\alpha_{i,1}(\varepsilon(\alpha_{i,0})+1)}\cdots A_i^{\alpha_{i,i}(\varepsilon(\alpha_{i,0})+1)+\delta(\alpha_{i,0})}A_{n-1}^{\alpha_{i,n-1}(\varepsilon(\alpha_{i,0})+1)}\rho_i^2.\end{aligned}$$

Comparing coefficients in K/L, for all $1\le i\le n-1$, we obtain:

$$\begin{aligned}&\beta_{1,i,k}+\cdots+\beta_{i-1,i,k}+\beta_{i,i+1,k}+\cdots+\beta_{i,n-1,k}+\beta_{i,n,k}=\alpha_{i,k}(\varepsilon(\alpha_{i,0})+1)\text{ for all }k\neq i\\ &\beta_{1,i,i}+\cdots+\beta_{i-1,i,i}+\beta_{i,i+1,i}+\cdots+\beta_{i,n-1,i}+\beta_{i,n,i}-1=\alpha_{i,i}(\varepsilon(\alpha_{i,0})+1)+\delta(\alpha_{i,0}).\end{aligned}\tag{10}$$

(d) Consider the relation $\underline{\rho_n^2=B_{1,n}\cdots B_{n-1,n}}$ in $P_n(\mathbb{R}P^2)$:

$$\begin{aligned}\overline{s}_*(B_{1,n}\cdots B_{n-1,n}) =&\rho_n^{2\alpha_{1,0}}A_1^{\beta_{1,n,1}}\cdots A_{n-1}^{\beta_{1,n,n-1}}B_{1,n}\cdots\rho_n^{2\alpha_{n-1,0}}A_1^{\beta_{n-1,n,1}}\cdots A_{n-1}^{\beta_{n-1,n,n-1}}B_{n-1,n}\\ =&\rho_n^{2(\alpha_{1,0}+\cdots+\alpha_{n-1,0})}A_1^{\beta_{1,n,1}+\cdots+\beta_{n-1,n,1}}\cdots A_{n-1}^{\beta_{1,n,n-1}+\cdots+\beta_{n-1,n,n-1}}\cdot\\ & B_{1,n}\cdots B_{n-1,n}\\ =&\rho_n^{2(\alpha_{1,0}+\cdots+\alpha_{n-1,0})}A_1^{\beta_{1,n,1}+\cdots+\beta_{n-1,n,1}}\cdots A_{n-1}^{\beta_{1,n,n-1}+\cdots+\beta_{n-1,n,n-1}}\cdot\\ & \rho_n^2\rho_{n+1}^{-2}A_1\cdots A_{n-1}\\ =&\rho_n^{2(\alpha_{1,0}+\cdots+\alpha_{n-1,0}-1)}A_1^{\beta_{1,n,1}+\cdots+\beta_{n-1,n,1}+1}\cdots A_{n-1}^{\beta_{1,n,n-1}+\cdots+\beta_{n-1,n,n-1}+1}\rho_n^2,\end{aligned}$$

using the relations $B_{1,n}\cdots B_{n-1,n}B_{n,n+1}=\rho_n^2$ and $A_1\cdots A_{n-1}B_{n,n+1}=\rho_{n+1}^2$, and the fact that $\rho_n^2\rightleftharpoons\rho_{n+1}^2$ in $P_{n+1}(\mathbb{R}P^2)/L$.

$$\begin{aligned}\overline{s}_*(\rho_n^2) =&\rho_{n+1}^{\alpha_{n,0}}A_1^{\alpha_{n,1}}\cdots A_{n-1}^{\alpha_{n,n-1}}\rho_n\cdot\rho_{n+1}^{\alpha_{n,0}}A_1^{\alpha_{n,1}}\cdots A_{n-1}^{\alpha_{n,n-1}}\rho_n\\ =&\rho_{n+1}^{\alpha_{n,0}}A_1^{\alpha_{n,1}}\cdots A_{n-1}^{\alpha_{n,n-1}}\rho_{n+1}^{-\alpha_{n,0}}A_1^{-\delta(\alpha_{n,0})}\cdots A_{n-1}^{-\delta(\alpha_{n,0})}A_1^{\alpha_{n,1}}\cdots A_{n-1}^{\alpha_{n,n-1}}\rho_n^2\\ =&A_1^{\varepsilon(\alpha_{n,0})\alpha_{n,1}}\cdots A_{n-1}^{\varepsilon(\alpha_{n,0})\alpha_{n,n-1}}A_1^{-\delta(\alpha_{n,0})}\cdots A_{n-1}^{-\delta(\alpha_{n,0})}A_1^{\alpha_{n,1}}\cdots A_{n-1}^{\alpha_{n,n-1}}\rho_n^2\\ =&A_1^{\alpha_{n,1}(\varepsilon(\alpha_{n,0})+1)-\delta(\alpha_{n,0})}\cdots A_{n-1}^{\alpha_{n,n-1}(\varepsilon(\alpha_{n,0})+1)-\delta(\alpha_{n,0})}\rho_n^2.\end{aligned}$$

Comparing coefficients in K/L, we obtain:

$$\begin{aligned}&\alpha_{1,0}+\cdots+\alpha_{n-1,0}=1\\ &\beta_{1,n,i}+\cdots+\beta_{n-1,n,i}+1=\alpha_{n,i}(\varepsilon(\alpha_{n,0})+1)-\delta(\alpha_{n,0})\text{ for all }i=1,\dots,n-1.\end{aligned}\tag{11}$$

Now consider equation (10) modulo 2. For all $1 \leq i \leq n-1$, we have:

$$\begin{aligned}\delta(\alpha_{i,0}) &\equiv \beta_{1,i,i} + \cdots + \beta_{i-1,i,i} + \beta_{i,i+1,i} + \cdots + \beta_{i,n-1,i} + \beta_{i,n,i} + 1 \\ &\equiv \delta(\alpha_{1,0}) + \cdots + \delta(\alpha_{i-1,0}) + \delta(\alpha_{i+1,0}) + \cdots + \delta(\alpha_{n-1,0}) + (\delta(\alpha_{i,0}) + \delta(\alpha_{n,0})) + 1,\end{aligned}$$

using equations (6), (7) and (9). Hence

$$\delta(\alpha_{i,0}) \equiv 1 + \sum_{j=1}^{n} \delta(\alpha_{j,0}) \pmod 2 \text{ for all } 1 \leq i \leq n-1, \tag{12}$$

and thus $\delta(\alpha_{1,0}) \equiv \cdots \equiv \delta(\alpha_{n-1,0}) \pmod 2$. Further, since $x \equiv \delta(x) \pmod 2$ for all $x \in \mathbb{Z}$, we see from equation (11) that $\sum_{j=1}^{n-1} \delta(\alpha_{j,0}) \equiv 1 \pmod 2$, $\delta(\alpha_{1,0}) \equiv \cdots \equiv \delta(\alpha_{n-1,0}) \equiv 1 \pmod 2$ and that n is even. It follows from equation (12) that $\delta(\alpha_{1,0}) \equiv \cdots \equiv \delta(\alpha_{n-1,0}) \equiv \delta(\alpha_{n,0}) \equiv 1 \pmod 2$, and so $\alpha_{1,0}, \ldots, \alpha_{n,0}$ are odd. Since n is even, the element $B_{2,3}$ exists. Further, $3 \leq n-1$, and hence $\beta_{2,3,0} = 0$ from equation (5). Now consider the image in $P_{n+1}(\mathbb{R}P^2)/L$ under $\overline{s}_*$ of the relation $\rho_1 \rightleftharpoons B_{2,3}$ of $P_n(\mathbb{R}P^2)$:

$$\begin{aligned}\overline{s}_*(\rho_1 B_{2,3}) &= \rho_{n+1}^{\alpha_{1,0}} A_1^{\alpha_{1,1}} \cdots A_{n-1}^{\alpha_{1,n-1}} \rho_1 \cdot A_1^{\beta_{2,3,1}} \cdots A_{n-1}^{\beta_{2,3,n-1}} B_{2,3} \\ &= \rho_{n+1}^{\alpha_{1,0}} A_1^{\alpha_{1,1}+\beta_{2,3,1}} \cdots A_{n-1}^{\alpha_{1,n-1}+\beta_{2,3,n-1}} \rho_1 B_{2,3}. \\ \overline{s}_*(B_{2,3}\rho_1) &= \rho_{n+1}^{\alpha_{1,0}} A_1^{\varepsilon(\alpha_{1,0})\beta_{2,3,1}+\alpha_{1,1}} \cdots A_{n-1}^{\varepsilon(\alpha_{1,0})\beta_{2,3,n-1}+\alpha_{1,n-1}} B_{2,3}\rho_1.\end{aligned}$$

Comparing coefficients in K/L, we see that $\beta_{2,3,i}(\varepsilon(\alpha_{1,0})-1) = 0$ for all $i = 1, \ldots, n-1$. Since $\alpha_{1,0}$ is odd, $\varepsilon(\alpha_{1,0}) = -1$, and thus $\beta_{2,3,i} = 0$ for all $i = 1, \ldots, n-1$. Hence $\beta_{2,3,2} = 0$. But since n is even, $3 \leq n-1$, and this contradicts equation (6). Hence $\overline{p}_*$ does not admit a section, and so neither does p_*. This proves the first statement of the theorem. The second statement follows from the fact that we mentioned in the introduction, that under the hypotheses of the theorem, the fibration $p\colon F_{n+m}(\mathbb{R}P^2) \longrightarrow F_n(\mathbb{R}P^2)$ admits a section if and only if the group homomorphism $p_*\colon P_{n+m}(\mathbb{R}P^2) \longrightarrow P_n(\mathbb{R}P^2)$ does. $\square$

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